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On the Control of Mechanical Manipulators[1]

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Abstract

The dynamic performance of computer-controlled manipulators is directly linked to the formulation of the dynamic model of manipulators and its corresponding control law. Several approaches are available in formulating the dynamic models of mechanical manipulators and most notably of these are the Lagrange-Euler formulation and the Newton-Euler formulation. This paper describes an efficient position plus derivative control in joint space for a PUMA robot arm whose dynamic equations of motion are formulated by the Newton-Euler method. The controller compensates the inertia loading, the nonlinear coupling reactance forces between joints and the gravity loading effects. Computer simulation of the performance of the control law is included for discussion.

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1. Introduction

The purpose of manipulator control is to maintain the dynamical response of an electromechanical manipulator in accordance with some pre-specified system performance and desired goals. In general, the control problem consists of (i) obtaining dynamic models of the physical system and (ii) specifying corresponding control laws or strategies to achieve the desired system response and performance. This paper deals with the latter part of the control problem of computer-controlled manipulators, and in particular, the PUMA robot arm.

A mechanical manipulator can be treated as an open-loop articulated chain with several rigid bodies (links) connected in series by either revolute or prismatic joints. One end of the chain is attached to a supporting base while the other end is free and attached with a tool (the end-effector) to manipulate objects or perform assembly tasks. The motion of the joints result in relative motion of the links. A priori information needed for control is a set of differential equations describing the dynamic behavior of the manipulator. Though various approaches are available to formulate the robot arm dynamics such as the Lagrange-Euler [Lew74], the "Recursive-Lagrange" [Hol80], the Newton-Euler [LWP80], and more recently the "Gibbs-Appell" [HoT80] formulation, two main approaches remained to be used by most researchers to systematically derive the dynamic model of the manipulator - the Lagrange-Euler and the Newton-Euler formulations. After obtaining the dynamic equations of motion of the manipulator, a suitable control law must be designed to compute the necessary torques/forces to actuate the joints for every set point ($\vartheta^d, \dot{\vartheta}^d, \ddot{\vartheta}^d$) in the pre-planned trajectory. Bejczy [Bej74] based on the Lagrangian formulation has shown that the dynamic equations of motion for a 6-jointed manipulator are highly nonlinear and consists of inertia loading, coupling reactance forces between joints and gravity loading effects. Hence, the control law must be designed to compensate all these nonlinear effects. A position plus derivative control based on the computed torque technique has been used previously to servo a Stanford arm [Mar73] whose dynamic equations of motion are formulated by applying the Lagrangian equations of motion to an open articulated chain. However, the dynamic equations of motion as formulated by the Lagrange-Euler method have been shown to be computationally inefficient [TML80,Pau72], and real-time control based on the 'complete' dynamic model has been found difficult to achieve if not impossible [Pau72]. A simple control law in joint space which compensates the inertia loading, the coupling reactance forces between joints and the gravity loading will be shown through the "Equivalence Formulation" [TML80] to have the same control effects as the one obtained by the computed torque technique. This control law is based on the dynamic equations of motion formulated by the Newton-Euler method. Computer simulation of the performance of the control law for a PUMA robot arm on a VAX-11/780 computer shows the expected result.

In the following sections, vectors are represented in boldface lower case alphabets while matrices are in boldface upper case alphabets.

2. Kinematics and Notation for Manipulators

A mechanical manipulator consists of a sequence of rigid bodies, called links, connected by either revolute or prismatic joints. Each pair of joint-link constitutes one degree of freedom. Hence for an n degree-of-freedom manipulator, there are n pairs of joint-link with link 0 attached to a supporting base where an inertial coordinate frame is established. In order to describe the translational and rotational relationship between adjacent links, a Denavit-Hartenberg matrix representation for a pair of joint-link is used [DeH55] and shown in Figure 1. From Figure 1, an orthonormal coordinate frame system (x_i, y_i, z_i) is assigned to the i^{th} pair of joint-link, where the z_i axis passes through the axis of motion of joint i+1, and the x_i axis is normal to the z_{i-1} axis, while the y_i axis completes the right hand rule. With this orthonormal coordinate frame, link i is characterized by two parameters: a_i, the common normal distance between z_{i-1} and z_i axes, and α_i, the twist angle measured between z_{i-1} and z_i axes; and joint i which connects link i-1 to link i is characterized by a distance parameter d_i measured between x_{i-1} and x_i axes and a joint variable ϑ_i if it is revolute. If joint i is prismatic, then it is characterized by an angle parameter ϑ_i and a joint variable d_i. With the coordinate frames established for adjacent links (link i and link i-l), one can relate the relationship between the adjacent coordinate frames (i^{th} and $i-1^{th}$ frames) by performing the following four operations (see Figure 1): (a) Rotate an angle of ϑ_i about z_{i-1} axis. (b) Translate a distance of d_i along z_i axis. (c) Translate a distance of a_i along rotated x_{i-1} axis. (d) Rotate an angle of α_i about the rotated x_i axis.

These four operations may be expressed as a chain product of four homogeneous coordinate transformation matrices [DeH55] as

$$T_{i-1}^{i} = \begin{bmatrix} \cos\vartheta_i & -\cos\alpha_i \sin\vartheta_i & \sin\alpha_i \sin\vartheta_i & a_i \cos\vartheta_i \\ \sin\vartheta_i & \cos\alpha_i \cos\vartheta_i & -\sin\alpha_i \cos\vartheta_i & a_i \sin\vartheta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

$$= \begin{bmatrix} A_{i-1}^{i} & p_{i-1}^{i} \\ 0 & 1 \end{bmatrix} \tag{2.2}$$

The upper left 3x3 matrix of T_{i-1}^{i} is called the rotation matrix A_{i-1}^{i} while the upper right 3x1 vector is called the position vector p_{i-1}^{i} . One can view the rotation matrix A_{i-1}^{i} as a transformation matrix which maps a vector $r_i = (x , y , z)^t$ expressed in the i^{th} coordinate frame into the $(i-1)^{th}$ coordinate frame with both origins coincided at one point, and the position vector as the displacement vector of the origin of the i^{th} coordinate frame from the origin of the $(i-1)^{th}$ coordinate frame.

Rotation matrices A_{i-1}^{i} have the following useful properties:

(i) The inverse matrix of A_{i-1}^{i} is its transpose.

$$[A_{i-1}^{i}]^{-1} = [A_{i-1}^{i}]^{t} = A_i^{i-1} \tag{2.3}$$

(ii) Rotations between coordinate frames i and j can be written as a chain product of rotation matrices between intermediate frames,

$$A_i^j = A_i^{i+1} A_{i+1}^{i+2} \cdots A_{j-1}^{j} \quad ; \text{for } i < j \tag{2.4}$$

$$= \prod_{k=i+1}^{j} A_i^k$$

(iii) The rotation matrix operates on the cross vector as :

$$A_{i-1}^{i}(r \times s) = (A_{i-1}^{i}\, r) \times (A_{i-1}^{i}\, s) \tag{2.5}$$

where r and s are vectors expressed in the i^{th} coordinate frame.

(iv) Since the rotation matrix A_o^i is a function of the joint variable ϑ_i or d_i, its partial derivative with respect to the joint variable q_j (where $q_j = \vartheta_j$) is given as:

$$\frac{\partial A_o^i}{\partial q_j} = \begin{cases} A_o^{j-1} Q_j^j A_{j-1}^i & ; \; j \le i; \; i=1,2,\ldots,n \\ 0 & ; \; j>i \end{cases} \tag{2.6}$$

where

$$Q_j^j = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \text{if } \; q_j \equiv \vartheta_j \tag{2.7}$$

Since

$$(A_j^i z_i) \times b_j = Q_j^i \, b_j \tag{2.8}$$

where z_i is a unit vector $(0,0,1)^t$ along the z_i axis (or the axis of motion) and $\mathbf{b}_j$ is any vector whose coordinates are expressed in the j^{th} coordinate frame, we can rewrite Eqn (2.6) as

$$\frac{\partial A_o^i}{\partial q_j} = A_o^{j-1} \, (z_j \times A_{j-1}^i) \tag{2.9}$$

Similarly one obtains the second partial derivative of A_o^i as:

$$\frac{\partial^2 A_o^i}{\partial q_j \partial q_k} = \begin{cases} A_o^{j-1} \, (z_{j-1} \times A_{j-1}^{k-1} \, (z_{k-1} \times A_{k-1}^i)) & ; \quad i \ge k \ge j \\ A_o^{k-1} \, (z_{k-1} \times A_{k-1}^{j-1} \, (z_{j-1} \times A_{j-1}^i)) & ; \quad i \ge j \ge k \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{2.10}$$

Let us define a displacement vector r_i fixed in link i as:

$$p_{i-1}^i = A_{i-1}^i \, r_i \tag{2.11}$$

where p_{i-1}^i is the displacement of the origin of the i^{th} coordinate frame from the origin of the $(i-1)^{th}$ coordinate frame. Using the summation of vectors, we obtain:

$$p_o^i = \sum_{k=1}^{i} A_o^k \; r_k \tag{2.12}$$

and

$$\frac{\partial p_o^i}{\partial q_j} = \sum_{k=1}^{i} A_o^{j-1} \; (z_{j-1} \times A_{j-1}^k \; r_k) \; ; \; j \le i \; ; \; i=1,2,\ldots n \tag{2.13}$$

$$\frac{\partial^2 p_o^i}{\partial q_j \partial q_k} = \sum_{m=1}^{i} A_o^{j-1} \, (z_{j-1} \times A_{j-1}^{k-1} \, (z_{k-1} \times A_{k-1}^m \, r_m)) \quad ; j \le k \tag{2.14}$$

The following vector identities will be needed for later use:

$$a \times (b \times c) = (a^t c) b - c (a^t b) = (\mathrm{Tr}(c\, a^t) U - c\, a^t) \, b \tag{2.15}$$

$$\mathrm{Tr}(a (b \times c)^t) = (b \times c)^t a = (c \times a)^t b = b^t (c \times a) \tag{2.16}$$

$$a \times (b \times c) = b \times (a \times c) + (a \times b) \times c \tag{2.17}$$

where U is an identity matrix of appropriate dimension.

Consider a differential mass dm in link i, if we integrate the outer product $r_i r_i^t$ over the entire mass, we obtain the link inertia matrix J_i^r about the pivot of link i defined by:

$$J_i^r = \int r_i r_i^t dm \tag{2.18}$$

or

$$J_i^r = \begin{bmatrix} \int r_x^2 dm & \int r_x r_y dm & \int r_x r_z dm \\ \int r_y r_x dm & \int r_y^2 dm & \int r_y r_z dm \\ \int r_z r_x dm & \int r_z r_y dm & \int r_z^2 dm \end{bmatrix} \tag{2.19}$$

Using the parallel axis theorem [Sym71], J_i^r can be rewritten in terms of the i^{th} link center of mass inertia matrix, I_i, and the center of mass vector, $\bar{r}_i$, as below:

$$J_i^r = \begin{bmatrix} \frac{-I_{xx}+I_{yy}+I_{zz}}{2} + \bar{r}_x^2 m_i & \bar{r}_x \bar{r}_y m_i & \bar{r}_x \bar{r}_z m_i \\ \bar{r}_y \bar{r}_x m_i & \frac{I_{xx}-I_{yy}+I_{zz}}{2} + \bar{r}_y^2 m_i & \bar{r}_y \bar{r}_z m_i \\ \bar{r}_z \bar{r}_x m_i & \bar{r}_z \bar{r}_y m_i & \frac{I_{xx}+I_{yy}-I_{zz}}{2} + \bar{r}_z^2 m_i \end{bmatrix} \tag{2.20}$$

Note that the inertial tensor, I_i, of link i can be written as:

$$I_i = (Tr(J_i^r - \bar{r}_i \bar{r}_i^t))U - (J_i^r - \bar{r}_i \bar{r}_i^t) \tag{2.21}$$

Finally, to be consistent with the dimensionality of the homogeneous transformation matrix T_{i-1}^i we define an augmented 4x4 matrix, J_i, which has the form:

$$J_i = \begin{bmatrix} J_i^r & m_i \bar{r}_i \\ m_i \bar{r}_i^t & m_i \end{bmatrix}, \tag{2.22}$$

where m_i is the mass of the i^{th} link.

The above kinematics relationship between adjacent links and their inertia will be used in the following sections to derive the dynamic equations of motion and show the equivalence of the two controllers based on the two most popular arm dynamics formulations.

3. Dynamics of Manipulators

The dynamic equations of motion for the PUMA robot arm can be obtained from known physical laws (Newtonian and Lagrangian mechanics) and physical measurements (link inertias and parameters). The actual derivation is based on the Lagrangian/Newtonian formulation applied to open articulated chains represented in Denavit-Hartenberg matrix notation form. The equations of motion for a six-jointed manipulator have been derived previously by Bejczy[Bej74], Paul[Pau72] and Lewis[Lew74] using Lagrangian generalized coordinates. The equations of motion derived from the Lagrangian and Newtonian formulations will be briefly presented here.

3.1. Lagrange-Euler Formulation [Lew74]

Consider a position vector expressed in homogeneous coordinates, $p = (x, y, z, 1)^t$, which points from the base coordinate system to a differential mass, dm, located in the i^{th} link. **p** can be written as:

$$\mathbf{p}=T_o^i\mathbf{r}_i\,, \quad \text{and} \quad \mathbf{r}_i = (x_i\,, y_i\,, z_i\,, 1)^t \tag{3.1}$$

where $\mathbf{r}_i$ is the position of the differential mass dm represented in the i^{th} coordinate frame independent of ϑ_i.

The velocity of this differential mass with respect to the base coordinate frame (an inertial frame) is:

$$v_o^i = \frac{dp}{dt} = \left[\sum_{j=1}^{i} \frac{\partial T_o^i}{\partial q_j}\dot{q}_j\right] r_i \;\; ; \text{ for } i=1,2,\ldots n \tag{3.2}$$

The associated kinetic energy dK_i is $\frac{1}{2}Tr(v_o^i(v_o^i)^t)dm$ which equals:

$$dK_i = \frac{1}{2}\sum_{j=1}^{i}\sum_{k=1}^{i} Tr\left\{\frac{\partial T_o^i}{\partial q_j} r_i(r_i)^t dm \left(\frac{\partial T_o^i}{\partial q_k}\right)^t \dot{q}_j\dot{q}_k\right\} \tag{3.3}$$

When each link is integrated over its entire mass and the kinetic energies of all links are summed, we have:

$$K.E. = \sum_{i=1}^{n}\int dK_i = \sum_{i=1}^{n}\left\{\frac{1}{2}Tr\left\{\sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial T_o^i}{\partial q_j} J_i \left(\frac{\partial T_o^i}{\partial q_k}\right)^t \dot{q}_j\dot{q}_k\right\}\right\} \tag{3.4}$$

J_i is defined as in Eqn. (2.22) and n=6 for a PUMA robot arm.

The total potential energy of the arm is the sum of the potential energy of each link expressed in the base coordinate frame:

$$P.E. = \sum_{i=1}^{6} P_i = \sum_{i=1}^{6} -m_i g T_o^i \bar{r}_i \tag{3.5}$$

where

$\bar{r}_i$ is the position of the center of mass of link i

g is the gravity vector $= (0,0,-|g|,0)^t$ and $g = 9.8 m/s^2$

Applying the Lagrange-Euler equations of motion to the Lagrangian L = K.E. − P.E., we obtain the necessary generalized torque τ_i for joint i to drive the i^{th} link of the arm:

$$\begin{aligned}\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= \sum_{k=i}^{6}\sum_{j=1}^{k} Tr\left\{\frac{\partial T_o^k}{\partial q_j} J_k \left(\frac{\partial T_o^k}{\partial q_i}\right)^t\right\}\ddot{q}_j \\ &+ \sum_{m=i}^{6}\sum_{j=1}^{m}\sum_{k=1}^{m} Tr\left\{\frac{\partial^2 T_o^m}{\partial q_j \partial q_k} J_m \left(\frac{\partial T_o^m}{\partial q_i}\right)^t\right\}\dot{q}_j\dot{q}_k - \sum_{j=i}^{6} m_j g \frac{\partial T_o^j}{\partial q_i}\bar{r}_j \;\; ; \text{ for } i=1,2,\ldots,6\end{aligned} \tag{3.6}$$

Because of its matrix structure, this formulation is appealing from a control viewpoint in that it gives a set of closed form differential equations as:

$$D(\vartheta)\ddot{\vartheta} + H(\vartheta,\dot{\vartheta}) + G(\vartheta) = \tau \tag{3.7}$$

where:

$D(\vartheta)$ = a 6x6 inertia acceleration matrix

$$= \sum_{k=i}^{6}\sum_{j=1}^{k} Tr\left\{\frac{\partial T_o^k}{\partial q_j} J_k \left(\frac{\partial T_o^k}{\partial q_i}\right)^t\right\} \;\; ; \text{ for } i=1,2,\ldots,6 \tag{3.8}$$

$H(\vartheta\,,\dot{\vartheta})$ = a 6x1 nonlinear Coriolis and Centrifugal vector

$$= \sum_{m=i}^{6} \sum_{j=1}^{m} \sum_{k=1}^{m} Tr\left\{ \frac{\partial^2 T_o^m}{\partial q_j \partial q_k} J_m \left(\frac{\partial T_o^m}{\partial q_i} \right)^t \right\} \dot{q}_j \dot{q}_k \tag{3.9}$$

$G(\vartheta)$ = a 6x1 gravity loading vector of the links

$$= \sum_{j=i}^{6} m_j \; g \frac{\partial T_o^j}{\partial q_i} \bar{r}_j \quad ; \text{for } i=1,2,\ldots,6 \tag{3.10}$$

$\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_6)^t$
$\dot{\vartheta} = (\dot{\vartheta}_1, \dot{\vartheta}_2, \cdots, \dot{\vartheta}_6)^t$
$\ddot{\vartheta} = (\ddot{\vartheta}_1, \ddot{\vartheta}_2, \cdots, \ddot{\vartheta}_6)^t$
$\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)^t$
= external applied torques for the joints

This form allows one to design a control law that compensates all these nonlinear effects easily. Computationally, however, the Lagrangian formulation is extremely inefficient as compared with other formulations.

3.2. Newton-Euler Formulation [LWP80]

The Newton-Euler equations of motion of a manipulator consist of a set of compact forward and backward recursive equations. The most significant of this formulation is the computation time of the applied torques could be reduced tremendously so that real-time control is possible. A brief derivation of the formulation based on [LWP80] is presented here for completeness.

In order to avoid pseudo-forces, the time derivatives of all vectors must be taken with respect to an inertial frame established in link 0 (the base coordinate frame). All vectors are represented with respect to the base coordinate frame.

The forward recursive equations propagate linear velocity, linear acceleration, angular velocity, angular acceleration, total link forces and moments from the base to the end-effector of the manipulator. For manipulators having all the rotary joints, these equations are:

$$\omega_i = \omega_{i-1} + \dot{\vartheta}_i z_{i-1} \tag{3.11}$$

$$\alpha_i = \alpha_{i-1} + \omega_{i-1} \times \dot{\vartheta}_i z_{i-1} + \ddot{\vartheta}_i z_{i-1} \tag{3.12}$$

$$a_i = \omega_i \times (\omega_i \times r_i) + \alpha_i \times r_i + a_{i-1} \tag{3.13}$$

$$\bar{a}_i = \omega_i \times (\omega_i \times \bar{r}_i) + \alpha_i \times \bar{r}_i + a_i \tag{3.14}$$

$$F_i = m_i \bar{a}_i \tag{3.15}$$

$$\begin{aligned} N_i &= \omega_i \times (I_i \omega_i) + I_i \alpha_i \\ &= n_i - n_{i+1} + \bar{r}_i \times f_{i+1} - (\bar{r}_i + r_i) \times f_i \\ &= n_i - n_{i+1} - (\bar{r} + r_i) \times m_i \bar{a}_i - r_i \times f_{i+1} \end{aligned} \tag{3.16}$$

The backward recursive equations of motion propagate, from the end-effector to the base of the manipulator, the forces and moments exerted on link i by link i-1.

$$f_i = F_i + f_{i+1} \tag{3.17}$$

$$n_i = I_i\alpha_i + \omega_i \times (I_i\omega_i) + (\bar{r}_i + r_i) \times m_i\bar{a}_i + r_i \times f_{i+1} + n_{i+1} \tag{3.18}$$

$$\tau_i = (z_{i-1})^t n_i \tag{3.19}$$

where:

ω_i = angular velocity of link i

α_i = angular acceleration of link i

r_i = the origin of the i^{th} frame with respect to the $i-1^{th}$ frame

$\bar{r}_i$ = center of mass of link i with respect to the i^{th} frame

a_i = linear acceleration of link i

$\bar{a}_i$ = linear acceleration of the center of mass of link i

I_i = inertia about center of mass of link i.

F_i = total external force exerted on link i

N_i = total moment exerted on link i

f_i = force exerted on link i by link i-1

n_i = moment exerted on link i by link i-1

τ_i = torque exerted on link i

Inertias I_i and vectors $\bar{r}_i$, r_i and z_{i-1} take on complex multi-angle dependent forms if they are expressed in the base coordinate frame as required by Eqns. (3.11)-(3.19). If they are expressed in their own i^{th} coordinate frame they are constant. We will apply a rotation A_o^i as defined in Eqn (2.2) to Eqns. (3.11)-(3.19) to allow all the vectors and tensors to be represented in their own coordinate frames. Take Eqn. (3.18) for an example. Under rotation it becomes:

$$\begin{aligned} A_i^o n_i = {} & A_i^o I_i A_o^i A_i^o \alpha_i + A_i^o \omega_i \times (A_i^o I_i A_o^i A_i^o \omega_i) \\ & + m_i(A_i^o \bar{r}_i + A_i^o r_i) \times A_i^o \bar{a}_i + A_i^o r_i \times A_i^{i+1} A_{i+1}^o f_{i+1} \\ & + A_i^{i+1} A_{i+1}^o n_{i+1} \end{aligned} \tag{3.20}$$

where $A_i^o I_i A_o^i$ and $A_i^o r_i$ are in the i^{th} coordinate frame and hence constant. Rather than carry this notation we will assume all vectors and tensors are defined in the coordinate frame of their subscript, i.e. we will represent $A_i^o \omega_i$ by ω_i and $A_i^o I_i A_o^i$ simply by I_i. With this notation, Eqns. (3.11)-(3.19) become:

$$\omega_i = A_i^{i-1}(\omega_{i-1} + \dot{\vartheta}_i z_{i-1}) \tag{3.21}$$

$$\alpha_i = A_i^{i-1}(\alpha_{i-1} + \omega_{i-1} \times \dot{\vartheta}_i z_{i-1} + \ddot{\vartheta}_i z_{i-1}) \tag{3.22}$$

$$a_i = \omega_i \times (\omega_i \times r_i) + \alpha_i \times r_i + A_i^{i-1} a_{i-1} \tag{3.23}$$

$$\bar{a}_i = \omega_i \times (\omega_i \times \bar{r}_i) + \alpha_i \times \bar{r}_i + a_i \tag{3.24}$$

$$f_i = m_i \bar{a}_i + A_i^{i+1} f_{i+1} = F_i + A_i^{i+1} f_{i+1} \tag{3.25}$$

$$\begin{aligned} n_i = {} & I_i\alpha_i + \omega_i \times (I_i\omega_i) + m_i(\bar{r}_i + r_i) \times \bar{a}_i + r_i \times A_i^{i+1} f_{i+1} \\ & + A_i^{i+1} n_{i+1} \end{aligned} \tag{3.26}$$

$$\tau_i = (A_i^{i-1} z_{i-1})^t n_i \tag{3.27}$$

with the initial conditions of $\omega_o = 0$, $a_o = g z_o$, $\alpha_o = 0$ and $g = 9.8 m/s^2$

4. Computed Torque Technique Based on the Newtonian Equations of Motion

Given the equations of motion of a manipulator as in Eqn. (3.6) (Lagrange-Euler formulation) or Eqns. (3.21)-(3.27) (Newton-Euler formulation), the control problem is to find appropriate torques/forces to servo all the joints of the manipulator in real-time to track a desired position trajectory as closely as possible. Several methods are available in accomplishing this task. Most notably of these are: (i) Resolved Motion Rate Control (RMRC)[Whi69], (ii) Cerebellar Model Articulation Controller (CMAC)[Alb75], (iii) Near-minimum-time control [KaB71], and (iv) Computed torque technique [Mar73, Pau72].

The RMRC is a technique for determining the joint angle rates required to cause a manipulator end point (or tool) to move in certain directions which are expressed in hand or world coordinate system. In order to find the required $\dot{\vartheta}$, the inverse Jacobian matrix $J(\vartheta)^{-1}$ is required. One of the drawbacks of this method is the added computation load needed to find the inverse Jacobian matrix and the singularity problem associated with the matrix inversion.

The CMAC is a table look-up control method which based on neuro-physiological theory. It computes control functions by referring to a table stored in the computer memory rather than by solution of analytic equations. For useful applications several problems such as memory size management and accuracy need to be solved.

Due to the nonlinearity and complexity of the dynamical model of manipulator, a closed form solution of the optimal control is very difficult, if not impossible. Near-minimum-time control is based on the linearization of the equations of motion about the nominal trajectory and linear feedback and/ or suboptimal control law are obtained analytically. This control method is still too complex to be used for manipulators with four or more degree of freedom and furthermore it neglects the effect of unknown external loads.

One of the basic control schemes is the computed torque technique [Mar73] based on the Lagrange-Euler equations of motion. This section presents an analogous control law derived from the computed torque technique based on the equations of motion derived from the Newton-Euler method as in Eqns (3.21)-(3.27).

The computed torque technique assumes that one can accurately compute the counterparts of $D(\vartheta)$, $H(\vartheta, \dot{\vartheta})$, and $G(\vartheta)$ in Eqn. (3.6) to minimize their nonlinear effects and use a position plus derivative control to servo the joints [Pau72]. Thus the structure of the control law has the form of:

$$\tau = D_a(\vartheta)\left[\ddot{\vartheta}^d + K_v(\dot{\vartheta}^d - \dot{\vartheta}) + K_p(\vartheta^d - \vartheta)\right] + H_a(\vartheta, \dot{\vartheta}) + G_a(\vartheta) \tag{4.1}$$

where

K_v is a 6x6 velocity feedback gain matrix.

K_p is a 6x6 position feedback gain matrix.

Substituting the τ from Eqn (4.1) into Eqn. (3.6), we have

$$D(\vartheta)\ddot{\vartheta} + H(\vartheta,\dot{\vartheta}) + G(\vartheta) = D_a(\vartheta)\left[\ddot{\vartheta}^d + K_v(\dot{\vartheta}^d - \dot{\vartheta}) + K_p(\vartheta^d - \vartheta)\right] + H_a(\vartheta, \dot{\vartheta}) + G_a(\vartheta) \tag{4.2}$$

If $D_a(\vartheta)$, $H_a(\vartheta, \dot{\vartheta})$, $G_a(\vartheta)$ are equal to $D(\vartheta)$, $H(\vartheta, \dot{\vartheta})$, $G(\vartheta)$ respectively, then Eqn 4.2 reduces to

$$D(\vartheta)\left[\ddot{e} + K_v\dot{e} + K_p e\right] = 0 \tag{4.3}$$

Since $D(\vartheta)$ is always non-singular, and if (a) K_v is symmetric and non-negative definite matrix, (b) K_p is symmetric and positive definite matrix, and (c) the rank of $\left[K_v \mid K_pK_v \mid \cdots \mid K_p^{n-1}K_v\right] = n$, then

$$\lim_{t\to\infty} e(t) \to 0$$

The analogous control law derived from the computed torque technique based on Eqns. (3.21)-(3.27) can be obtained by substituting $\ddot{\vartheta}_i$ in these equations with

$$\ddot{\vartheta}_i^d + \sum_{s=1}^{n} K_v^{is}(\dot{\vartheta}_s^d - \dot{\vartheta}_s) + \sum_{s=1}^{n} K_p^{is}(\vartheta_s^d - \vartheta_s)$$

or

$$\ddot{\vartheta}_i^d + \sum_{s=1}^{n} K_v^{is}\dot{e}_s + \sum_{s=1}^{n} K_p^{is}e_s \qquad (4.4)$$

where K_v^{is} and K_p^{is} are derivative and position feedback gains for joint i respectively.

The following derivation shows that the recursive equations of motion with the above substitution of $\ddot{\vartheta}_i$ will give the desired position plus derivative control for a PUMA robot arm.

Eqn. (3.21) can be expanded to obtain:

$$\omega_i=\sum_{j=1}^{i}(A_i^{j-1}z_{j-1})\dot{\vartheta}_j \qquad (4.5)$$

And Eqn. (3.22) expanded as:

$$\alpha_i=\sum_{j=1}^{i}(A_i^{j-1}z_{j-1})(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js}e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i-1}(A_i^{k-1}z_{k-1}) \times (A_i^{j-1}z_{j-1})\dot{\vartheta}_j\dot{\vartheta}_k \qquad (4.6)$$

With Eqn. (2.8), the first term in Eqn. (3.23) becomes

$$\omega_i \times (\omega_i \times r_i) = \sum_{j=1}^{i}\sum_{k=1}^{i}(A_i^{j-1}z_{j-1}) \times ((A_i^{k-1}z_{k-1}) \times r_i))\dot{\vartheta}_j\dot{\vartheta}_k = \sum_{j=1}^{i}\sum_{k=1}^{i} Q_i^{j-1}Q_i^{k-1}r_i\dot{\vartheta}_j\dot{\vartheta}_k \qquad (4.7)$$

With the help of Eqn. (2.17) and Eqn. (2.8), the second term of Eqn. (3.23) becomes

$$\begin{aligned}\alpha_i \times r_i &= \sum_{j=1}^{i}(A_i^{j-1}z_{j-1}) \times r_i(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js}e_s) \\ &+ \sum_{j=1}^{i}\sum_{k=1}^{i-1}((A_i^{k-1}z_{k-1}) \times (A_i^{j-1}z_{j-1} \times r_i)-(A_i^{j-1}z_{j-1}) \times ((A_i^{k-1}z_{k-1}) \times r_i))\dot{\vartheta}_j\dot{\vartheta}_k \\ &= \sum_{j=1}^{i} Q_i^{j-1}r_i(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js}e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i-1}(Q_i^{k-1}Q_i^{j-1}-Q_i^{j-1}Q_i^{k-1})r_i\dot{\vartheta}_j\dot{\vartheta}_k\end{aligned} \qquad (4.8)$$

Combining Eqn. (4.7) and Eqn. (4.8) and expanding the a_{i-1} term in Eqn. (3.23), we have:

$$a_i=\sum_{m\geq j}^{i}\sum_{j=1}^{i}Q_i^{j-1}A_i^m r_m(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s)$$
$$+\sum_{m\geq max(j,k)}^{i}\sum_{j=1}^{i}\sum_{k=1}^{i}Q_i^{v-1}Q_i^{u-1}A_i^m r_m\dot{\vartheta}_j\dot{\vartheta}_k+A_i^o g z_o \quad (4.9)$$

where u=max(j,k) and v=min(j,k)

Using Eqn. (2.17), and Eqn. (2.8), this becomes:

$$a_i=\sum_{m\geq j}^{i}\sum_{j=1}^{m}A_i^{j-1}Q_{j-1}^{j-1}A_{j-1}^m r_m(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s)$$
$$+\sum_{m\geq max(j,k)}^{i}\sum_{j=1}^{m}\sum_{k=1}^{m}A_i^{k-1}Q_{k-1}^{k-1}A_{k-1}^{j-1}Q_{j-1}^{j-1}A_{j-1}^m r_m\dot{\vartheta}_j\dot{\vartheta}_k \quad (4.10)$$

From Eqn. (2.13) and (2.14) we have:

$$a_i=A_i^o\sum_{j=1}^{i}\sum_{k=1}^{i}(\frac{\partial p_o^i}{\partial\vartheta_j}(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s)+\frac{\partial p_o^i}{\partial\vartheta_j\partial\vartheta_k}\dot{\vartheta}_k\dot{\vartheta}_j)+A_i^o g z_o=A_i^o\frac{d^2p_o^i}{dt^2} \quad (4.11)$$

As expected, the i^{th} acceleration with respect to the base rotated into the i^{th} coordinate frame.

$\bar{a}_i$ in Eqn. (3.24) can be written as:

$$\bar{a}_i=\sum_{j=1}^{i}A_i^{j-1}A_{j-1}^{j-1}A_{j-1}^i\bar{r}_i(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s)$$
$$+\sum_{j=1}^{i}\sum_{k=1}^{i}A_i^{k-1}Q_{k-1}^{k-1}A_{k-1}^{j-1}Q_{j-1}^{j-1}A_{j-1}^i\bar{r}_i\dot{\vartheta}_j\dot{\vartheta}_k+a_i\ , \quad (4.12)$$

which using Eqns. (2.6)-(2.10) becomes:

$$\bar{a}_i=A_i^o\sum_{j=1}^{i}(\frac{\partial A_o^i}{\partial\vartheta_j}\bar{r}_i+\frac{\partial p_o^i}{\partial\vartheta_j})(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s)$$
$$+A_i^o\sum_{j=1}^{i}\sum_{k=1}^{i}(\frac{\partial^2A_o^i}{\partial\vartheta_j\partial\vartheta_k}\bar{r}_i+\frac{\partial p_o^i}{\partial\vartheta_j\partial\vartheta_k})\dot{\vartheta}_j\dot{\vartheta}_k+A_i^o g z_o\ . \quad (4.13)$$

For the present we will ignore the last two terms of Eqn. (3.26), which represent force and moment exerted by the hand (end-effector) on the object. We will discuss their contributions later.

Eqn. (3.27) then simplifies to:

$$\tau_i=(I_i\alpha_i+\omega_i\times(I_i\omega_i)+m_i(r_i+\bar{r}_i)\times\bar{a}_i)A_i^{i-1}z_{i-1}\quad ;\ i=1,2,..n \quad (4.14)$$

Using Eqn. (2.21) and Eqn. (2.15) this is:

$$\tau_i=((Tr(J_i^r)U-J_i^r)\alpha_i+\omega_i\times((Tr(J_i^r)U-J_i^r)\omega_i)-\bar{r}_i\times(\alpha_i\times\bar{r}_i)$$
$$-\bar{r}_i\times(\omega_i\times(\omega_i\times\bar{r}_i))+m_i(\bar{r}_i+r_i)\times\bar{a}_i)A_i^{i-1}z_{i-1} \quad (4.15)$$

Consider the J_i^r terms of Eqn. (4.15) in the light of Eqn. (2.18). If the integral over mass is pulled outside of the trace, these terms become:

$$\int((Tr(r_ir_i^t)U-r_ir_i^t)\alpha_iA_i^{i-1}z_{i-1}$$
$$+\omega_i\times((Tr(r_ir_i^t)U-r_ir_i^t)\omega_i)A_i^{i-1}z_{i-1})dm\ , \quad (4.16)$$

using Eqn. (2.15):

$$=\int \left(r_i \times (\alpha_i \times r_i) A_i^{i-1} z_{i-1} + \omega_i \times (r_i \times (\omega_i \times r_i)) A_i^{i-1} z_{i-1}\right) dm$$
$$=\int \left(((\alpha_i \times r_i)(A_i^{i-1} z_{i-1} \times r_i) + \omega_i \times (\omega_i \times r_i)(A_i^{i-1} z_{i-1} \times r_i)\right) dm$$
$$=\int \mathrm{Tr}\left\{(\alpha_i \times r_i)(Q_i^{i-1} r_i)^t + \omega_i \times (\omega_i \times r_i)(Q_i^{i-1} r_i)^t\right\} dm \tag{4.17}$$
$$=\int \mathrm{Tr}\left\{(\alpha_i \times r_i) r_i^t (Q_i^{i-1})^t + (\omega_i \times (\omega_i \times r_i)) r_i^t (Q_i^{i-1})^t\right\} dm$$

Using steps identical to Eqns. (4.5)-(4.8), Eqn. (4.17) becomes:

$$=\int \mathrm{Tr}\Big\{A_i^o \sum_{j=1}^{i} A_o^{j-1} Q_{j-1}^{j-1} A_{j-1}^{i} (\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js} \dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s)$$
$$+ \sum_{j=1}^{i}\sum_{k=1}^{i} (A_o^{k-1} A_{k-1}^{k-1} Q_{k-1}^{k-1} A_{k-1}^{j-1} Q_{j-1}^{j-1} A_{j-1}^{i}) r_i r_i^t (A_i^o A_o^{i-1} A_{i-1}^{i-1} A_i^{i-1})^t \Big\} dm \tag{4.18}$$

Bringing the integral over dm inside the trace operator and using Eqns. (2.6)-(2.10) we have:

$$\mathrm{Tr}\left\{A_i^o (\sum_{j=1}^{i} \frac{\partial A_o^i}{\partial \vartheta_j}(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial^2 A_o^i}{\partial \vartheta_j \partial \vartheta_k} \dot{\vartheta}_j \dot{\vartheta}_k) J_i^r \left[\frac{\partial A_o^i}{\partial \vartheta_i}\right]^t A_o^i\right\} \tag{4.19}$$

Since Tr{BCD}=Tr{CDB} and $A_i^o A_o^i = U$ the identity matrix, this is:

$$\sum_{j=1}^{i} \mathrm{Tr}\left\{\frac{\partial A_o^i}{\partial \vartheta_j} J_i^r \left[\frac{\partial A_o^i}{\partial \vartheta_i}\right]^t\right\}(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i} \mathrm{Tr}\left\{\frac{\partial^2 A_o^i}{\partial \vartheta_j \partial \vartheta_k} J_i^r \left[\frac{\partial A_o^i}{\partial \vartheta_i}\right]^t\right\} \dot{\vartheta}_j \dot{\vartheta}_k \tag{4.20}$$

Now consider the last three terms of Eqn. (4.15) and use Eqn. (4.13) it becomes:

$$\left[A_i^{i-1} z_{i-1}\right]^t \Big[m_i(\bar{r}_i + r_i) \times A_i^o (\sum_{j=1}^{i} \frac{\partial p_o^i}{\partial \vartheta_j}(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial^2 p_o^i}{\partial \vartheta_j \partial \vartheta_k} \dot{\vartheta}_j \dot{\vartheta}_k)$$
$$+ m_i r_i \times A_i^o (\sum_{j=1}^{i} \frac{\partial A_o^i}{\partial \vartheta_j}(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s)\bar{r}_i + \sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial^2 A_o^i}{\partial \vartheta_j \partial \vartheta_k} \dot{\vartheta}_j \dot{\vartheta}_k r_i)\Big] \tag{4.21}$$
$$+ \left[A_i^{i-1} z_{i-1}\right]^t \left[m_i(\bar{r}_i + r_i) \times A_i^o m_i g z_o\right]$$

Consider the first and third terms of Eqn. (4.21):

$$\left[A_i^{i-1} z_{i-1}\right]^t \left[m_i \bar{r}_i \times A_i^o \left[\sum_{j=1}^{i} \frac{\partial p_o^i}{\partial \vartheta_j}(\ddot{\vartheta}_j^d + \sum_{s=1}^{n} K_v^{js}\dot{e}_s + \sum_{s=1}^{n} K_p^{js} e_s) + \sum_{j=1}^{i}\sum_{k=1}^{i} \frac{\partial^2 p_o^i}{\partial \vartheta_j \partial \vartheta_k} \dot{\vartheta}_j \dot{\vartheta}_k\right]\right] \tag{4.22}$$

Using Eqn. (2.15) and Eqn. (2.8) this becomes:

$$\mathrm{Tr}\left\{A_i^o \left[\cdots\right] (A_{i-1}^{i} z_{i-1} \times m_i \bar{r}_i)^t\right\} = \mathrm{Tr}\left\{A_i^o \left[\cdots\right] m_i \bar{r}_i^t (Q_i^{i-1})^t\right\} \tag{4.23}$$

and since

$$Q_i^{i-1} = A_i^o A_o^{i-1} Q_{i-1}^{i-1} A_{i-1}^{i} = A_i^o \frac{\partial A_o^i}{\partial \vartheta_i} \tag{4.24}$$

Eqn. (4.23) becomes:

$$=Tr\left\{\left[\sum_{j=1}^{i}\frac{\partial p_o^i}{\partial\vartheta_j}+\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial^2 p_o^i}{\partial\vartheta_j\partial\vartheta_k}\dot\vartheta_j\dot\vartheta_k\right]m_i r_i^t\left(\frac{\partial A_o^i}{\partial\vartheta_i}\right)^t(\ddot\vartheta_j^d+\sum_{s=1}^{n}K_v^{js}\dot e_s+\sum_{s=1}^{n}K_p^{js}e_s)\right\} \tag{4.25}$$

Consider the next two terms of Eqn. (4.21).

$$\left(A_i^{i-1}z_{i-1}\right)^t m_i r_i\times A_i^o\left[\left(\sum_{j=1}^{i}\frac{\partial p_o^i}{\partial\vartheta_j}(\ddot\vartheta_j^d+\sum_{s=1}^{n}K_v^{js}\dot e_s+\sum_{s=1}^{n}K_p^{js}e_s)+\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial^2 p_o^i}{\partial\vartheta_j\partial\vartheta_k}\dot\vartheta_j\dot\vartheta_k\right)\right.$$
$$\left.+\left(\sum_{j=1}^{i}\frac{\partial A_o^i}{\partial\vartheta_j}(\ddot\vartheta_j^d+\sum_{s=1}^{n}K_v^{js}\dot e_s+\sum_{s=1}^{n}K_p^{js}e_s)\bar r_i+\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial^2 A_o^i}{\partial\vartheta_j\partial\vartheta_k}\dot\vartheta_j\dot\vartheta_k r_i\right)\right] \tag{4.26}$$

Using Eqn. (2.15) they can be recast into:

$$Tr\left\{A_i^o\left[\cdots\right](A_{i-1}^i z_{i-1}\times m_i r_i)^t\right\}\ , \tag{4.27}$$

and since by Eqn. (2.8) and Eqn (2.13):

$$A_{i-1}^i z_{i-1}\times r_i=Q_{i-1}^i r_i=A_i^o A_o^{i-1}Q_{i-1}^{i-1}A_{i-1}^i r_i=A_i^o\frac{\partial p_o^i}{\partial\vartheta_i} \tag{4.28}$$

they become:

$$Tr\left\{m_i A_i^o\left[\cdots\right]\left(\frac{\partial p_o^i}{\partial\vartheta_i}\right)^t\right\} \tag{4.29}$$

Consider the last term in Eqn. (4.21) and exchanging the dot product and vector product, it becomes

$$\left[A_i^{i-1}z_{i-1}\times m_i(\bar r_i+r_i)\right]^t A_i^o m_i g z_o$$
$$=(Q_i^{i-1}(\bar r_i+r_i))^t A_i^o m_i g z_o \tag{4.30}$$
$$=Tr\left[A_i^o m_i g z_o\left[Q_i^{i-1}(\bar r_i+r_i)\right]^t\right]$$

Using Eqn. (4.24) and Eqn. (4.28), it becomes

$$Tr\left[m_i g z_o\left[\frac{\partial A_o^i}{\partial\vartheta_i}\bar r_i+\frac{\partial p_o^i}{\partial\vartheta_i}\right]^t\right]$$
$$=Tr\left\{\left[\frac{\partial A_o^i}{\partial\vartheta_i}\bar r_i+\frac{\partial p_o^i}{\partial\vartheta_i}\right](m_i g z_o)^t\right\} \tag{4.31x}$$

Expanding it to homogeneous coordinates representation,

$$=m_i g\frac{\partial T_o^i}{\partial\vartheta_i}\bar r_i \tag{4.32}$$

where

$g=(0\ ,\ 0\ ,\ |g|\ ,\ 0\ )^t\quad ;\quad \bar r_i=(\bar x_i\ ,\ \bar y_i\ ,\ \bar z_i\ ,\ 1)^t$

Combining this result with Eqns. (4.25) and (4.29), we have:

$$\tau_i=\sum_{j=1}^{i}\mathrm{Tr}\left\{\left(\frac{\partial A_o^i}{\partial\vartheta_j}J_i^r+m_i\frac{\partial p_o^i}{\partial\vartheta_j}\bar{r}_i^t\right)\left[\frac{\partial A_o^i}{\partial\vartheta_i}\right]^t+\left(m_i\frac{\partial A_o^i}{\partial\vartheta_j}\bar{r}_i+m_i\frac{\partial p_o^i}{\partial\vartheta_j}\right)\left[\frac{\partial p_o^i}{\partial\vartheta_i}\right]^t\right\}\left(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\right)$$
$$+\mathrm{Tr}\left\{\sum_{j=1}^{i}\sum_{k=1}^{i}\left[\left(\frac{\partial^2 A_o^i}{\partial\vartheta_j\partial\vartheta_k}J_i^r+m_i\frac{\partial^2 p_o^i}{\partial\vartheta_j\partial\vartheta_k}\bar{r}_i^t\right)\left[\frac{\partial A_o^i}{\partial\vartheta_i}\right]^t\right.\right. \tag{4.33}$$
$$\left.\left.+\left(\frac{\partial^2 A_o^i}{\partial\vartheta_j\partial\vartheta_k}m_i\bar{r}_i+\frac{\partial^2 p_o^i}{\partial\vartheta_j\partial\vartheta_k m_i}\right)\left[\frac{\partial p_o^i}{\partial\vartheta_i}\right]^t\right]\right\}\dot{\vartheta}_j\dot{\vartheta}_k+m_i g\frac{\partial T_o^i}{\partial\vartheta_i}\bar{r}_i$$

which is equivalent to, if expanding into homogeneous coordinates,

$$\tau_i=\sum_{j=1}^{i}\mathrm{Tr}\left\{\frac{\partial T_o^i}{\partial\vartheta_j}J_i\left[\frac{\partial T_o^i}{\partial\vartheta_i}\right]^t\right\}\left(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\right)$$
$$+\sum_{j=1}^{i}\sum_{k=1}^{i}\mathrm{Tr}\left\{\frac{\partial^2 T_o^i}{\partial\vartheta_j\partial\vartheta_k}J_i\left[\frac{\partial T_o^i}{\partial\vartheta_i}\right]^t\right\}\dot{\vartheta}_k\dot{\vartheta}_j+m_i g\frac{\partial T_o^i}{\partial\vartheta_i}\bar{r}_i \tag{4.34}$$

where J_i is defined as in Eqn. (2.22).

This is the same result as Eqn. (3.6) when upper link contributions are ignored.

Consider now the upper link contributions, i.e. the last two terms in Eqn. (3.26):

$$\left\{A_i^{i-1}z_{i-1}\right\}^t\left(r_i\times A_i^{i+1}f_{i+1}+A_i^{i+1}n_{i+1}\right) \tag{4.35}$$

Assume for simplicity that $i+1$ is the last link of the arm, i.e. f_{i+2} and n_{i+2} are zero. We will relax this assumption shortly.

We can rewrite the first term of Eqn. (4.35) using Eqn. (2.8), and Eqn. (2.16) as:

$$\mathrm{Tr}\left\{A_i^{i+1}f_{i+1}(Q_i^{i-1}r_i)^t\right\}=\mathrm{Tr}\left\{A_o^{i+1}f_{i+1}(A_o^{i-1}Q_{i-1}^{i-1}r_i)^t\right\} . \tag{4.36}$$

Using Eqn. (3.25) and Eqn. (4.13) this can be written:

$$=\mathrm{Tr}\left\{\left[\sum_{j=1}^{i+1}\left(\frac{\partial A_o^{i+1}}{\partial\vartheta_j}\bar{r}_{i+1}+\frac{\partial p_o^{i+1}}{\partial\vartheta_j}\right)\left(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\right)\right.\right.$$
$$\left.\left.+\sum_{j=1}^{i+1}\sum_{k=1}^{i+1}\left(\frac{\partial^2 A_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}\bar{r}_{i+1}+\frac{\partial p_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}\right)\dot{\vartheta}_j\dot{\vartheta}_k+m_{i+1}gz_o\right](A_o^{i-1}Q_{i-1}^{i-1}r_i)^t\right\} \tag{4.37}$$

Using the same steps used in deriving Eqn. (4.33) the second term in Eqn. (4.35) $A_i^{i+1}n_{i+1}A_i^{i-1}z_{i-1}$, can be written:

$$
\begin{aligned}
&=\sum_{j=1}^{i+1}\mathrm{Tr}\Bigg\{\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_j}J_{i+1}^{r}+m_{i+1}\frac{\partial p_o^{i+1}}{\partial\vartheta_j}\bar{r}_{i+1}^{t}+m_{i+1}gz_c\Big)\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_i}\Big)^t \\
&\quad+\Big(m_{i+1}\frac{\partial A_o^{i+1}}{\partial\vartheta_j}\bar{r}_{i+1}+m_{i+1}\frac{\partial p_o^{i+1}}{\partial\vartheta_j}+m_{i+1}gz_c\Big)\big(A_o^{i-1}Q_{i-1}^{i-1}A_i^{i+1}r_{i+1}\big)^t\Bigg\}\cdot \\
&\quad\Big(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\Big) \qquad (4.38)\\
&\quad+\mathrm{Tr}\Bigg\{\sum_{j=1}^{i}\sum_{k=1}^{i}\Big(\Big(\frac{\partial^2 A_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}J_{i+1}^{r}+m_{i+1}\frac{\partial^2 p_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}\bar{r}_{i+1}^{t}\Big)\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_i}\Big)^t \\
&\quad+\Big(\frac{\partial^2 A_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}m_{i+1}\bar{r}_{i+1}+\frac{\partial^2 p_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}m_{i+1}\Big)\big(A_o^{i-1}Q_{i-1}^{i-1}A_i^{i+1}r_{i+1}\big)^t\Bigg\}\dot{\vartheta}_j\dot{\vartheta}_k\,,
\end{aligned}
$$

Since

$$
A_o^{i-1}Q_{i-1}^{i-1}r_i+A_o^{i-1}Q_{i-1}^{i-1}A_i^{i+1}r_{i+1}=\frac{\partial p_o^{i+1}}{\partial\vartheta_i}. \qquad (4.39)
$$

Eqn. (4.37) and Eqn. (4.38) can be combined to form:

$$
\begin{aligned}
&=\sum_{j=1}^{i+1}\mathrm{Tr}\Bigg\{\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_j}J_{i+1}^{r}+m_{i+1}\frac{\partial p_o^{i+1}}{\partial\vartheta_j}\bar{r}_{i+1}^{t}\Big)\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_i}\Big)^t+\Big(m_{i+1}\frac{\partial A_o^{i+1}}{\partial\vartheta_j}\bar{r}_{i+1}+m_{i+1}\frac{\partial p_o^{i+1}}{\partial\vartheta_j}\Big)\Big(\frac{\partial p_o^{i+1}}{\partial\vartheta_i}\Big)^t\Bigg\} \\
&\quad\Big(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\Big) \\
&\quad+\mathrm{Tr}\Bigg\{\sum_{j=1}^{i+1}\sum_{k=1}^{i+1}\Big(\Big(\frac{\partial^2 A_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}J_{i+1}^{r}+m_{i+1}\frac{\partial^2 p_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}\bar{r}_{i+1}^{t}\Big)\Big(\frac{\partial A_o^{i+1}}{\partial\vartheta_i}\Big)^t \qquad (4.40)\\
&\quad+\Big(\frac{\partial^2 A_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}m_{i+1}\bar{r}_{i+1}+\frac{\partial^2 p_o^{i+1}}{\partial\vartheta_j\partial\vartheta_k}m_{i+1}\Big)\Big(\frac{\partial p_o^{i+1}}{\partial\vartheta_i}\Big)^t\Big)\Bigg\}\dot{\vartheta}_j\dot{\vartheta}_k \\
&\quad+\mathrm{Tr}\Bigg[m_{i+1}gz_c\Bigg(\frac{\partial A_o^{i+1}}{\partial\vartheta_i}+\frac{\partial p_o^{i+1}}{\partial\vartheta_i}\Bigg)^t\Bigg]
\end{aligned}
$$

which when combined with Eqn. (4.34) can be reformed into:

$$
\begin{aligned}
\tau_i&=\sum_{q=i}^{i+1}\sum_{j=1}^{q}\mathrm{Tr}\Bigg\{\frac{\partial T_o^q}{\partial\vartheta_j}J_q\Bigg(\frac{\partial T_o^q}{\partial\vartheta_i}\Bigg)^t\Bigg\}\Big(\ddot{\vartheta}_j^d+\sum_{s=1}^{n}K_v^{js}\dot{e}_s+\sum_{s=1}^{n}K_p^{js}e_s\Big) \\
&\quad+\sum_{q=i}^{i+1}\sum_{j=1}^{q}\sum_{k=1}^{q}\mathrm{Tr}\Bigg\{\frac{\partial^2 T_o^q}{\partial\vartheta_j\partial\vartheta_k}J_q\Bigg(\frac{\partial T_o^q}{\partial\vartheta_i}\Bigg)^t\Bigg\}\dot{\vartheta}_k\dot{\vartheta}_j+\sum_{q=i}^{i+1}m_q g\frac{\partial T_o^q}{\partial\vartheta_i}\bar{r}_q
\end{aligned} \qquad (4.41)
$$

If the arm consists of n links then q can be summed from i to n, and we have Eqn. (4.1) which is the analogous control law derived from the computed torque technique based on the Lagrangian equations of motion.

5. Computer Simulation Results

This section discusses the computer simulation result of the proposed control law derived from the computed torque technique and compares its computational complexity with the analogous control law obtained from the Lagrange-Euler equations of motion. As a mean of comparison, their efficiency is determined based on the number of mathematical operations (multiplications and additions) in terms of the number of joints of the robot arm , n. The number of mathematical operations of some of the terms in both control laws may be slightly different from other papers [TML80,Hol80] due to the method of implementation of the control algorithms in programming.

In this study, the homogeneous transformation matrices T^i_{i-1} are computed first and then other relevant terms such as velocity, acceleration and gravity terms in the Lagrange-Euler equations of motion are computed respectively. The comparisons of the number of mathematical operations in the control laws based on these two formulations are tabulated in Table 1. and Table 2. In general, for a six-jointed robot with rotary joints, the number of mathematical operations in the control law as in Eqn (4.1) based on the Lagrangian formulation is about 100 times more than that of Newton-Euler formulation.

The feedback gains K_v and K_p of the control law are kept constant for the whole motion execution to facilitate the comparison of both control laws. The elements of K_v and K_p are assigned according to the stability criterion as outlined after Eqn. (4.3). Since it is unlikely that the natural frequency of a PUMA robot arm will be over ten hertz, the principal diagonal elements of K_p are assigned the value of 100 and the diagonal elements of K_v to $2\sqrt{K_p} = 20$ Again to simplify the comparison, all the non-diagonal elements of K_v and K_p are zero which neglect the position and derivative error effects between joints. Future investigation will focus on finding proper algorithms for selecting the elements of K_v and K_p.

Based on a PDP 11/45 computer and its manufacturer's specification sheet, an ADD (integer addition) instruction requires 300 ns and a MUL (integer multiply) instruction requires 3.3 μ-sec. If we assume that for each ADD and MUL instruction, we need to fetch data from the core memory and the memory cycle time is 450 ns, then the proposed control law based on the Newton-Euler formulation of robot arm dynamics requires approximately 3 msec. to compute the necessary torques/forces to servo all the joints of a PUMA robot arm for a position set point. This certainly is quite acceptable for the time delay in the servo loop and thus allows one to perform real-time control on a PUMA robot arm with all its dynamics taken into consideration.

A computer simulation study to evaluate the performance of the above control law for a PUMA robot, whose equations of motion are derived by the Newton-Euler formulation, was carried out on a VAX-11/780 computer. The initial and final joint angles of a PUMA 600 series robot arm are:

$$\vartheta_{\text{initial}} = (90^\circ , 0^\circ , 90^\circ , 0^\circ , 0^\circ , 0^\circ)^t$$

$$\vartheta_{\text{final}} = (45^\circ , 30^\circ , 0^\circ , 45^\circ , 60^\circ , 90^\circ)^t$$

The position errors $\vartheta^d(t) - \vartheta_a(t)$ for each joint are plotted and shown in Figures 3-8. The sampling time is chosen to be 0.01 second. Figure 2 shows the flow-chart for the computer simulation program implementation. In Figures 3-8, although the position errors are slightly "oscillatory" about the desired position set points, they are always well below and within the resolution of a 12-bit A/D converter (less than $4.26\text{x}10^{-4}$ radian per bit). Since the manipulator is a highly nonlinear and complex system, further improvements in the performance of the control law can be done by using adaptive feedback gains. Our future work will focus on finding proper adaptive control strategies for industrial robots whose loads are varying within a task cycle time.

Controller based on Lagrange-Euler Equations of Motion	Multiplications	Additions
T_j^i	$32n(n-1)$	$24n(n-1)$
$-m_j g \frac{\partial T_0^j}{\partial q_i} \bar{r}_j$	$4n(9n-7)$	$n\frac{(51n-45)}{2}$
$\sum_{j=i}^{n} m_j g \frac{\partial T_0^j}{\partial q_i} \bar{r}_j$	0	$\frac{1}{2}n(n-1)$
$Tr\left\{\frac{\partial T_0^k}{\partial q_j} J_k \left(\frac{\partial T_0^k}{\partial q_i}\right)^t\right\}$	$\frac{128}{3}n(n+1)(n+2)$	$\frac{65}{2}n(n+1)(n+2)$
$\sum_{k=max(i,j)}^{n} Tr\left\{\frac{\partial T_0^k}{\partial q_j} J_k \left(\frac{\partial T_0^k}{\partial q_i}\right)^t\right\}$	0	$\frac{1}{6}n(n-1)(n+1)$
$Tr\left\{\frac{\partial^2 T_0^m}{\partial q_j \partial q_k} J_m \left(\frac{\partial T_0^m}{\partial q_i}\right)^t\right\}$	$\frac{128}{3}n^2(n+1)(n+2)$	$\frac{65}{2}n^2(n+1)(n+2)$
$\sum_{m=max(i,j,k)}^{n} Tr\left\{\frac{\partial^2 T_0^m}{\partial q_j \partial q_k} J_m \left(\frac{\partial T_0^m}{\partial q_i}\right)^t\right\}$	0	$\frac{1}{6}n^2(n-1)(n+1)$
$\ddot{\vartheta}^d + K_v \dot{e} + K_p e$	$2n$	$4n$
$\tau = D_a(\ddot{\vartheta}^d + K_v \dot{e} + K_p e) + H_a(\dot{\vartheta}, \vartheta) + G_a(\vartheta)$	$n^2(n+2)$	$n^2(n+1)$
Total Mathematical Operations	$\frac{128}{3}n^4 + \frac{515}{3}n^3 + \frac{850}{3}n^2 + \frac{82}{3}n$	$\frac{98}{3}n^4 + \frac{787}{6}n^3 + \frac{640}{3}n^2 + \frac{131}{6}n$

where n = number of degree-of-freedom of the robot arm

Table 1 Breakdown of Mathematical Operations of the Controller Based on Lagrange-Euler Formulation

Controller based on Newton-Euler Equations of Motion	Multiplications	Additions
ω_i	9n	7n
α_i	9n	9n
a_i	27n	22n
$\bar{a}_i$	15n	14n
F_i	3n	0
f_i	9(n−1)	9n−6
N_i	24n	18n
n_i	21n−15	24n−15
$\ddot{\vartheta}_i^d + K_v\dot{e}_i + K_p e_i$	2n	4n
Total Mathematical Operations	119n−24	107n−21

where n = number of degree-of-freedom of the robot arm

Table 2 Breakdown of Mathematical Operations of the Controller Based on Newton-Euler Formulation

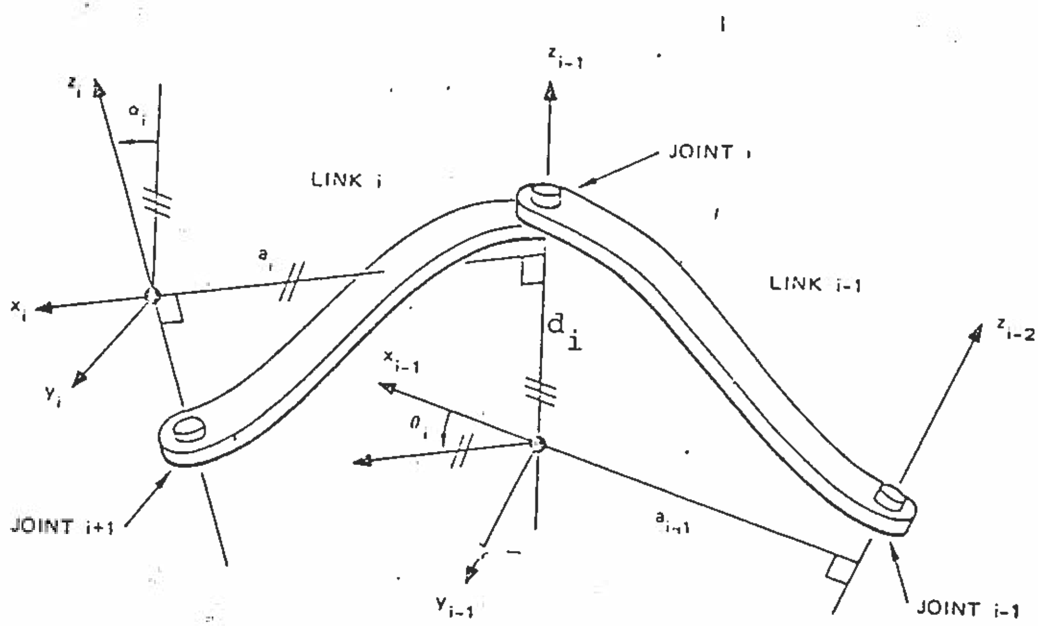
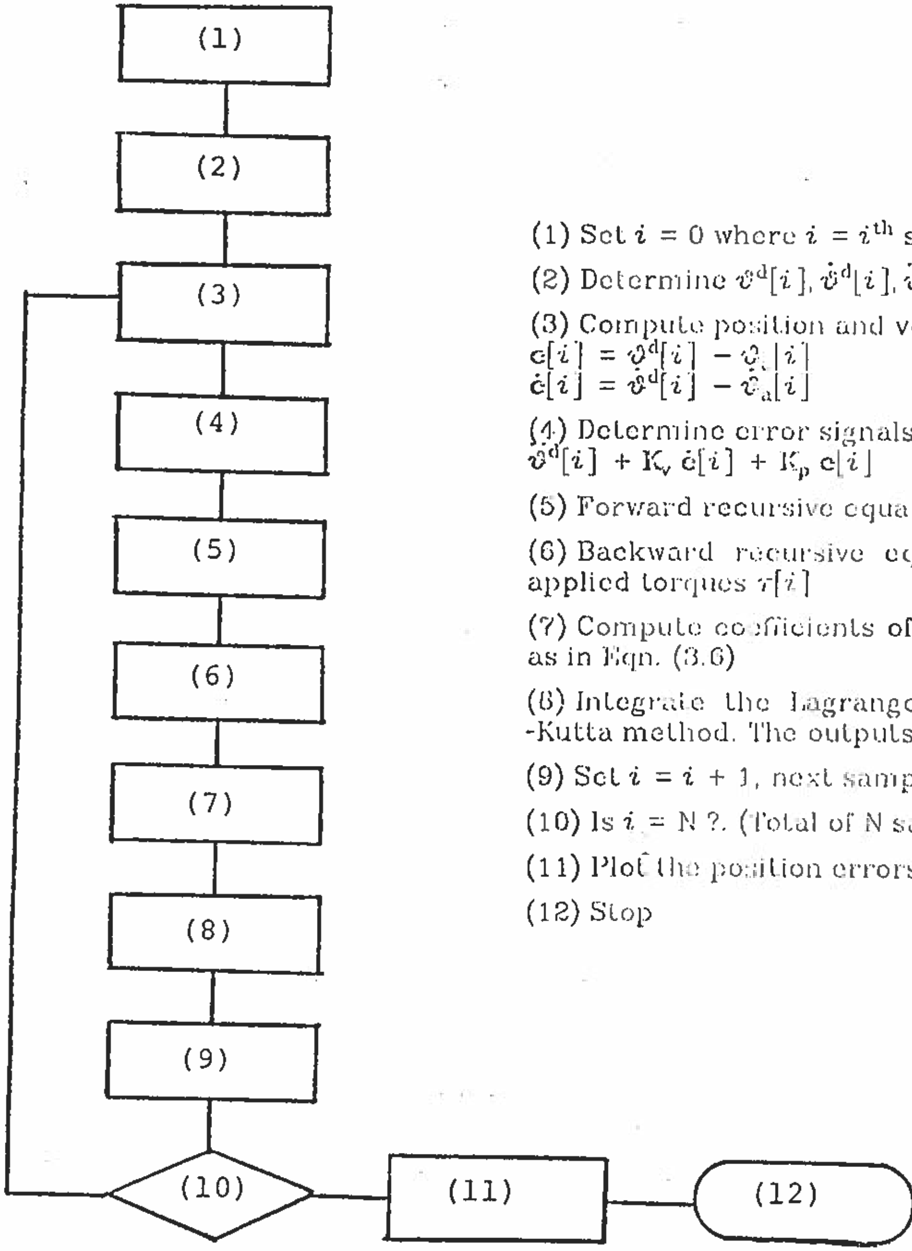


Figure 1 Parameters of A Coordinate System



(1) Set $i = 0$ where $i = i^{th}$ sampling period.

(2) Determine $\vartheta^d[i], \dot{\vartheta}^d[i], \ddot{\vartheta}^d[i]$ from preplanned trajectory.

(3) Compute position and velocity errors
$e[i] = \vartheta^d[i] - \vartheta_a[i]$
$\dot{e}[i] = \dot{\vartheta}^d[i] - \dot{\vartheta}_a[i]$

(4) Determine error signals with K_v, K_p feedback gain matrices.
$\ddot{\vartheta}^d[i] + K_v \dot{e}[i] + K_p e[i]$

(5) Forward recursive equations in Newton-Euler formulation

(6) Backward recursive equations in Newton-Euler formulation to determine applied torques $\tau[i]$

(7) Compute coefficients of robot arm model using Lagrange- Euler formulation as in Eqn. (3.6)

(8) Integrate the Lagrange-Euler equations of motion using 4^{th} order Runge -Kutta method. The outputs are $\vartheta_a[i]$ and $\dot{\vartheta}_a[i]$

(9) Set $i = i + 1$, next sampling period.

(10) Is $i = N$?. (Total of N sample periods).

(11) Plot the position errors $e_i = \vartheta_i^d - \vartheta_i^a$ for i=1,...,6

(12) Stop

Figure 2 Flow-Chart of Computer Simulation for the Proposed Controller

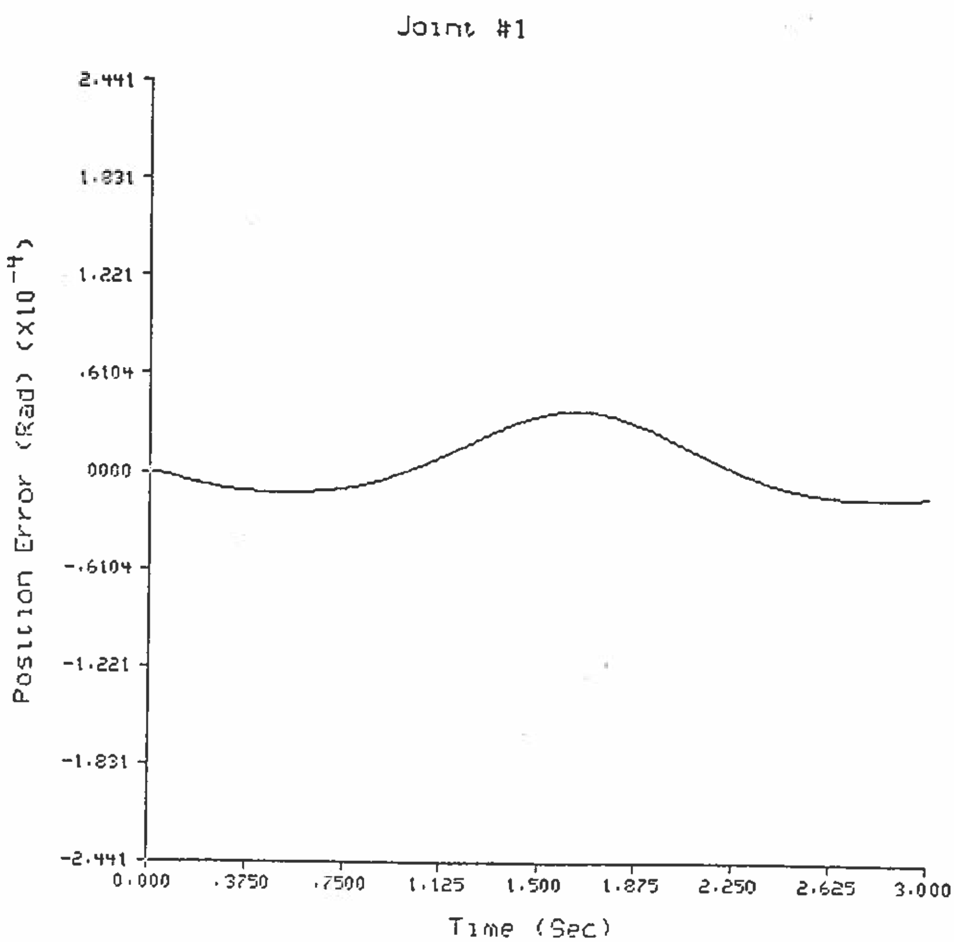


Figure 8 Position Error for Joint One

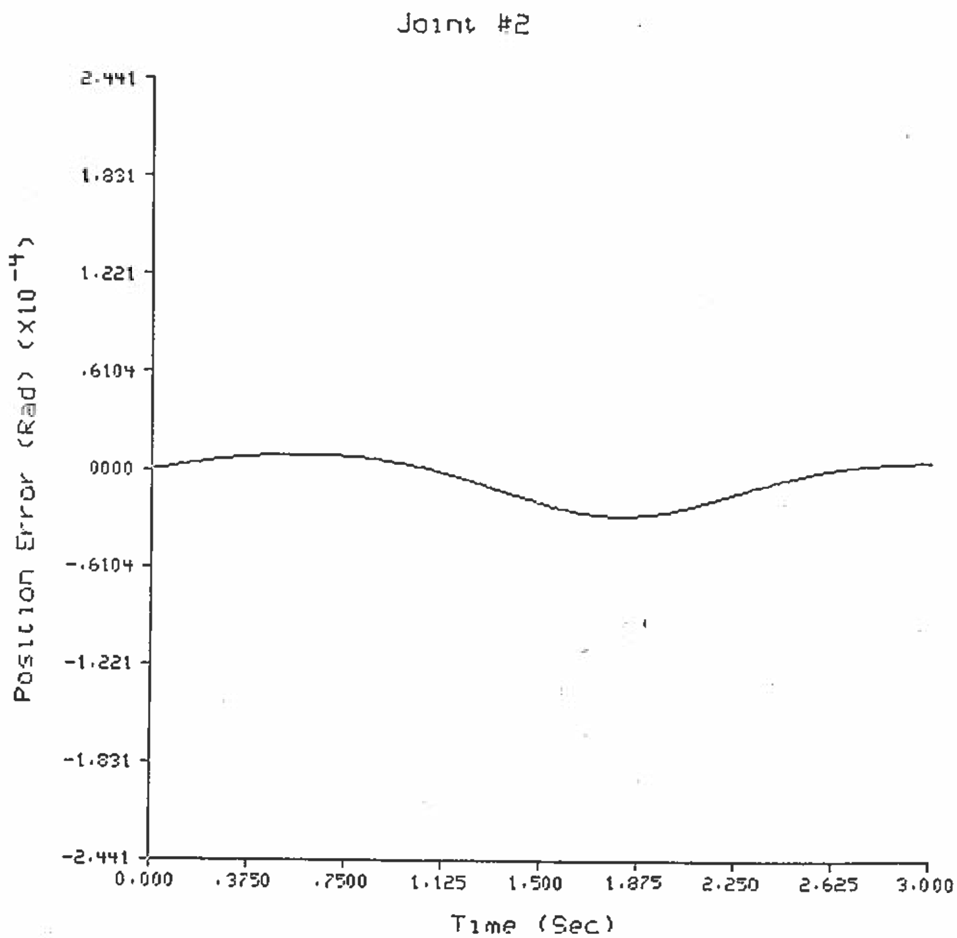


Figure 4 Position Error for Joint Two

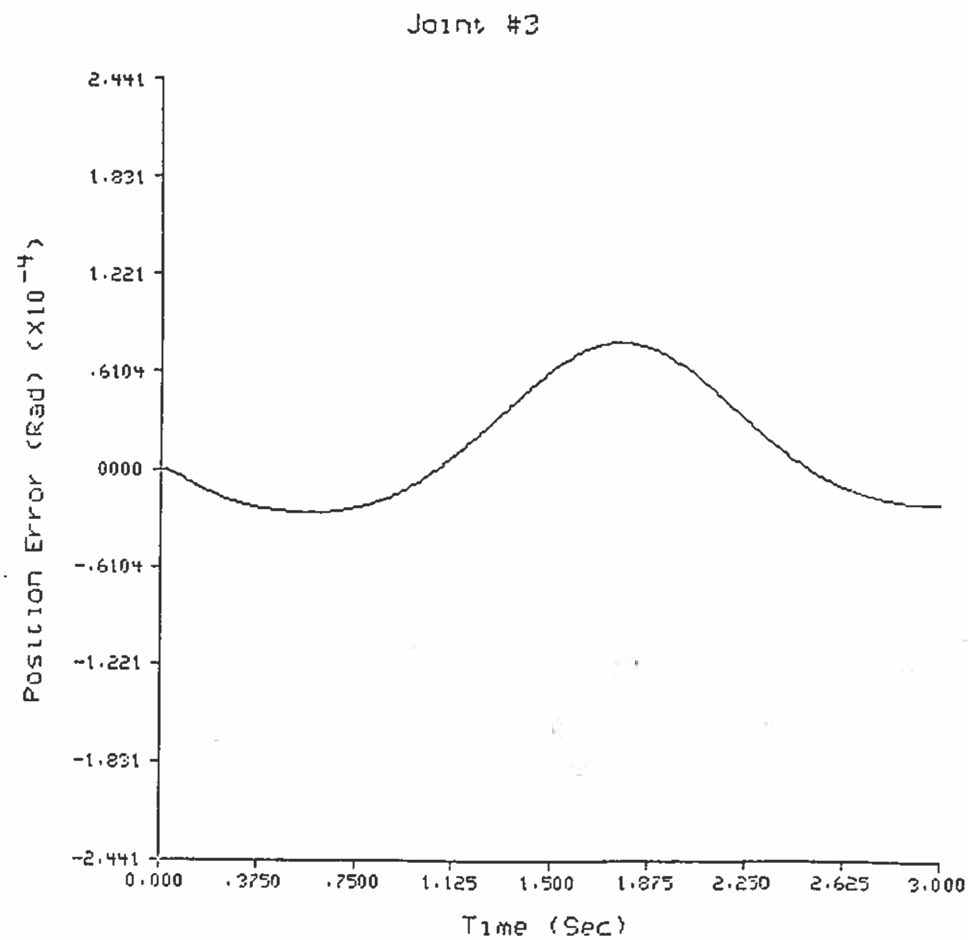


Figure 5 Position Error for Joint Three

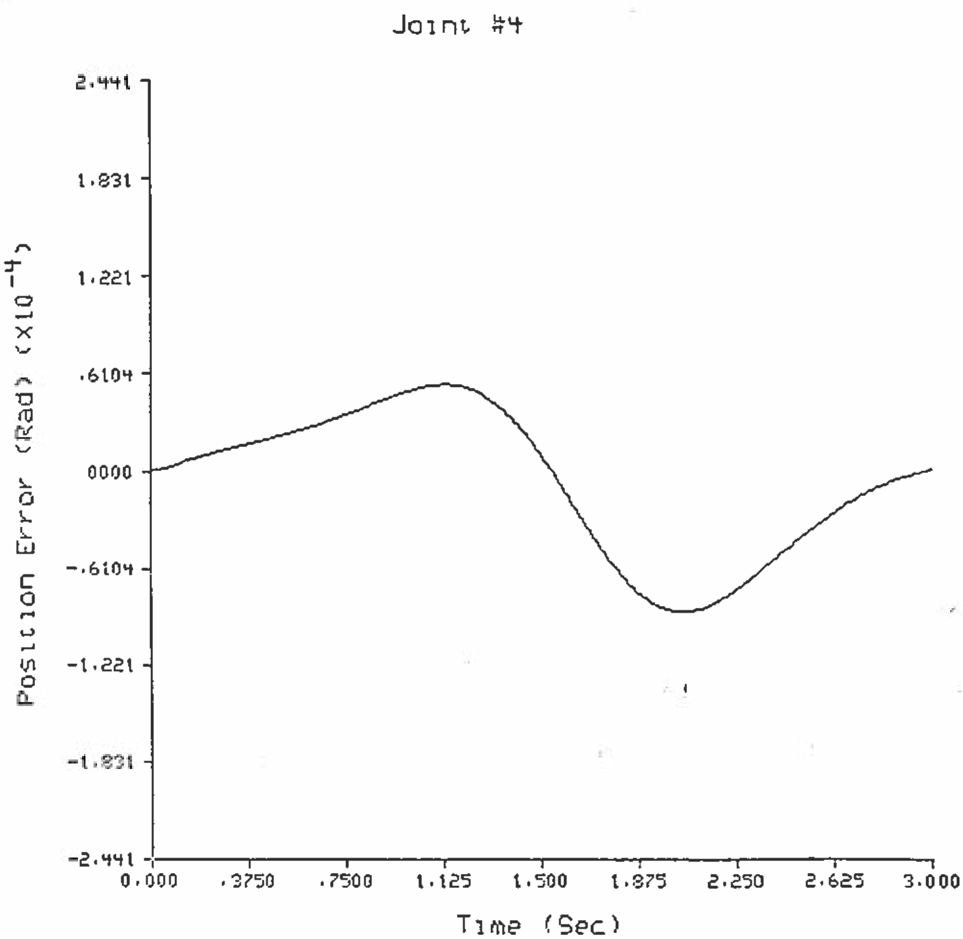


Figure 6 Position Error for Joint Four

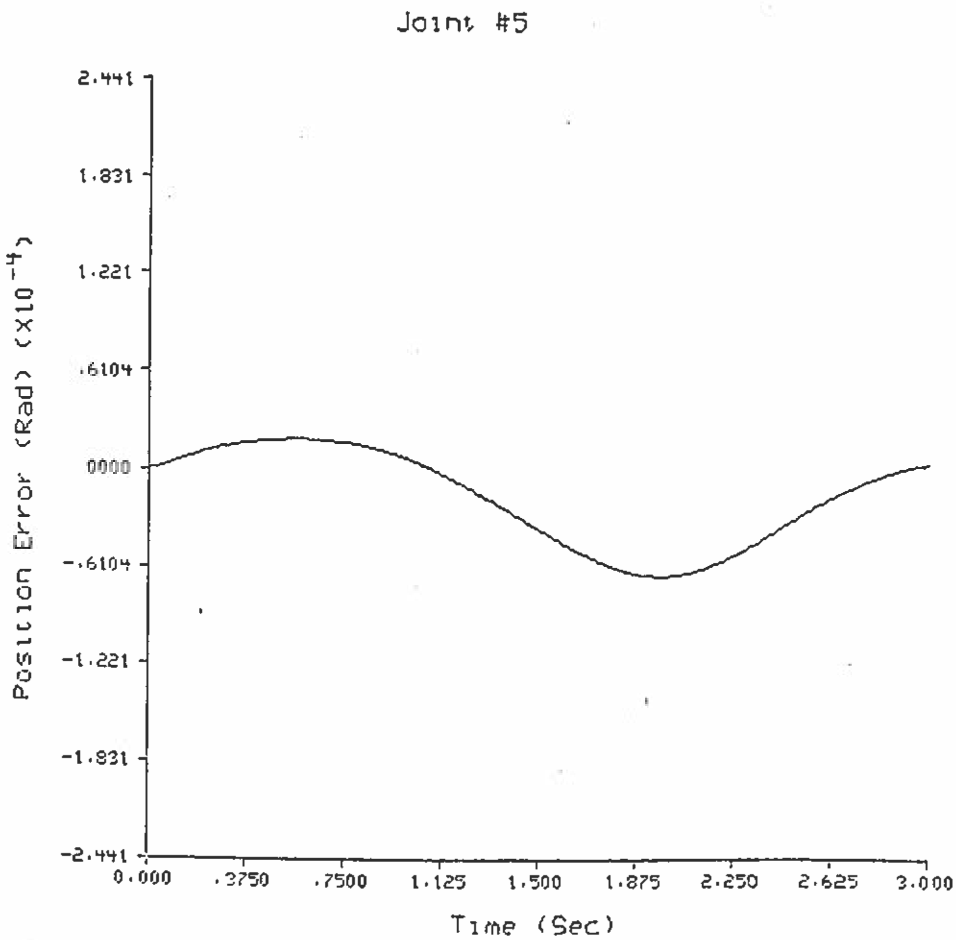


Figure 7 Position Error for Joint Five

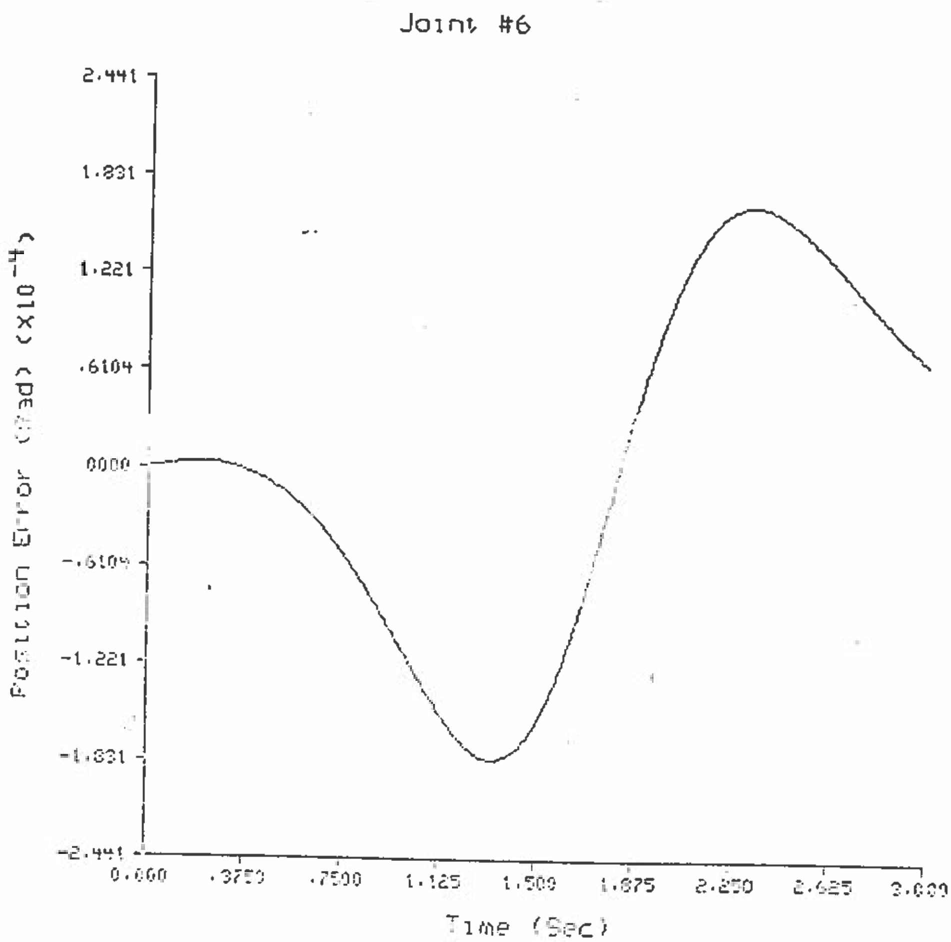


Figure 8 Position Error for Joint Six

6. Reference

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